Advances in Symplectic Tracking for Robust Modeling of Nonlinear Integrable Optics

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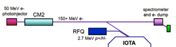
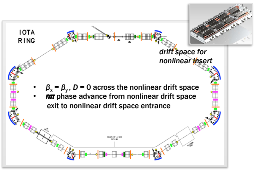
Nonlinear Integrable Optics in IOTA (FNAL)

Goals: Proof of principle demonstration of nonlinear integrable beam optics [1-3]--to achieve nonlinear tune shifts exceeding 0.25 without degradation of dynamic aperture and to exploit the resulting nonlinear decoherence to suppress collective instabilities (halo) [4].



- **Nonlinearity:** tune spread washes out coherent space charge instabilities
- **Integrability:** ensures orbits are regular and remain bounded (avoids chaos)



- **Dynamics inside the nonlinear magnetic insert:**

The vector potential is chosen to give a 2 d.o.f. Hamiltonian for on-energy particles within the insert of the form:

$$H_\perp = \frac{1}{2}(P_x^2+P_y^2) - \frac{\tau c^2}{\beta(s)}U\left(\frac{X}{c\sqrt{\beta(s)}},\frac{Y}{c\sqrt{\beta(s)}}\right)$$

Courant-Snyder transformation, scaling

$$H_N = \frac{1}{2}(P_{xN}^2+P_{yN}^2+X_N^2+Y_N^2) - \tau U(X_N,Y_N)$$

first invariant

D&N give in [1] a realizable potential U such that H_N admits a second invariant I_N: $\{H_N, I_N\}=0$.

- **Dynamics in the arc external to the insert:**

Assumed linear with a map R_N given by: $R_N = \pm I$ corresponding to a thin axially-symmetric lens generating a phase advance $n\pi$, for integer n.

H_N, I_N are invariant under the one-turn map for the ring.

Modeling need: Algorithms for robust and efficient symplectic long-term tracking with intense, high-resolution space charge in strongly nonlinear s-dependent $\boldsymbol{B}$ fields.

Complex Representation of the Nonlinear Integrable Potential

Integrability + Maxwell's equations require that the 2D magnetic vector potential $\mathcal{A}_s$ within the insert satisfies:

$$(\partial_x^2+\partial_y^2)\mathcal{A}_s = 0 \quad \textit{2D Laplace Eq.}$$

Bertrand-Darboux Eq.

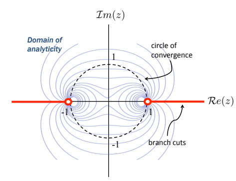
$$xy(\partial_x^2-\partial_y^2)\mathcal{A}_s + (y^2-x^2+1)\partial_x\partial_y\mathcal{A}_s + 3y\partial_x\mathcal{A}_s - 3x\partial_y\mathcal{A}_s = 0$$

$$z = x+iy \qquad F = \mathcal{A}_s + i\psi$$

Complex representation as a single ODE [5]:

$$(z^2-1)\frac{d^2F}{dz^2} + 3z\frac{dF}{dz} + 2\tilde{t} = 0$$

Solution: $F(z) = \left(\frac{\tilde{t}z}{\sqrt{1-z^2}}\right)\arcsin(z)$



Physical quantities

$$\vec{B} = \nabla\times\vec{A} = -\nabla\psi$$

$$F = \frac{A_z + i\psi}{B\rho}$$

$$\tilde{t} = \frac{\tau c^2}{\beta} \qquad z = \frac{x+iy}{c\sqrt{\beta}}$$

Domain of analyticity of the complex function F, which defines the vector potential of the nonlinear insert in the transverse plane. Curves in blue denote magnetic field lines. The fields vary with longitudinal position s due to the dependence on β, which is the betatron amplitude across the drift space containing the insert.

This representation is well-behaved and avoids a numerical instability associated with tracking using the form in [1].

Symplectic Integrator & Implementation in IMPACT-Z

Inside the nonlinear insert, tracking with space charge is performed using a second-order symplectic integrator (easily generalized to higher order) [7]. Twice apply the following splitting for a step of size h:

$$H = H_1 + H_2 \qquad \mathcal{M}(h) = \mathcal{M}_1\left(\frac{h}{2}\right)\mathcal{M}_2(h)\mathcal{M}_1\left(\frac{h}{2}\right) + O(h^3)$$

Upper-level splitting (once per "space charge" step)

$$H = H_{\text{ext}} + H_{SC} \qquad \text{(contribution due to space charge fields)}$$

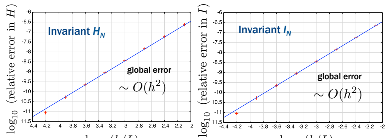
Lower-level splitting (once per "map" substep)

$$H_{\text{ext}} = H_{\text{drift}} + H_{NLL} \qquad \text{(contribution due to nonlinear insert fields)}$$

The map $\mathcal{M}_{SC}$ is a momentum kick in the space charge charge fields computed via a Poisson solve, while $\mathcal{M}_{NLL}$ is a momentum kick in the nonlinear insert fields using:

$$p^* = \sigma\frac{dF}{dz} \qquad p = \frac{\Delta p_x + i\Delta p_y}{p^0} \qquad \sigma = \frac{h}{c\sqrt{\beta}}$$

Convergence of the two invariants of motion with step size h



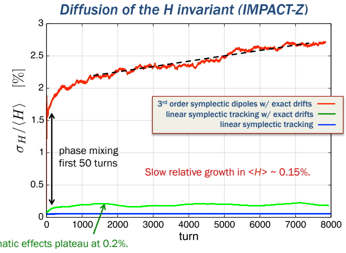
Error in the quantities H_N and I_N at the exit of the nonlinear insert obtained when tracking a single 2.5 MeV proton across the insert with parameters $\tau = 0.45$, $c = 0.01$ m$^{1/2}$, $\mu_0 = 0.3$, and $L = 2.0$ m.

Application to the IOTA lattice

As an example, we studied how undesired nonlinear effects in the arc (kinematic nonlinear effects and 2nd/3rd-order dipole effects) impact preservation of the invariants H_N and I_N in the IOTA ring.

2 nonlinear inserts ON	2.5 MeV protons
L=1.8 m, τ=0.45, c = 0.009 m$^{1/2}$, μ_0=0.3	Matched "nonlinear KV" beam ε_0 = 3.9 mm-mrad, σ_δ = 0

Diffusion of the H invariant (IMPACT-Z)



Initially, $\sigma_H \approx 0$. Note the rapid initial jump followed by linear growth. The corresponding value of $\sigma_I/\langle I\rangle$ grows by 1% over 8000 turns. Simulations using different values of ε_0 suggest that $\sigma_H/\langle H\rangle \sim \sqrt{\epsilon_0}$.

Tests of a 2D Symplectic Spectral Space Charge Solver

Following [8], a particle-based 2D symplectic space charge solver with rectangular boundary conditions was implemented in IMPACT-Z. The collective Hamiltonian for the N-particle system is given explicitly by:

$$H = \sum_{j=1}^{N_p} \underbrace{H_{\text{ext}}(\mathbf{r}_j,\mathbf{p}_j)}_{\text{single-particle Hamiltonian w/o space charge ("external")}} + \frac{K}{2}\sum_{i=1}^{N_p}\sum_{j=1}^{N_p} \underbrace{G(\mathbf{r}_i,\mathbf{r}_j)}_{\text{2D space charge Green function in a rectangular conducting pipe}}$$

Spectral approximation of G using N_l, N_m Fourier modes in x and y, respectively:

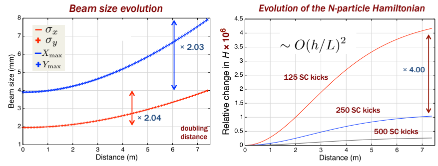
$$G(\mathbf{r}_i,\mathbf{r}_j) = 4\pi\frac{4}{ab}\frac{1}{N_p}\sum_{l=1}^{N_l}\sum_{m=1}^{N_p}\frac{1}{\gamma_{lm}^2}\sin(\alpha_l x_j)\sin(\beta_m y_j)\sin(\alpha_l x_i)\sin(\beta_m y_i)$$

for mode (l,m): $\alpha_l = \frac{l\pi}{a}, \quad \beta_m = \frac{m\pi}{b}, \quad \gamma_{lm}^2 = \alpha_l^2 + \beta_m^2$

Here K is the generalized perveance of the beam and a, b are the aperture size in x and y, respectively.

Benchmark: Expansion of a zero-emittance uniform cylinder beam

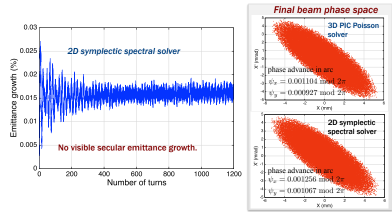
Simulation of a 2.5 MeV p beam with 4.113 mA current and initial radius 3.9 mm expanding in free space ($a = b = 5$ cm).



(1.024M particles, 512x512 modes) In free space, the N-particle Hamiltonian is an invariant of motion, which is numerically well-preserved.

Benchmark: A waterbag beam in the linear IOTA lattice (insert off)

An initially matched waterbag 2.5 MeV p beam with 0.411 mA and 8.0 mm-mrad (geometric) emittance is tracked in the linearized IOTA lattice for space charge tune depression 0.03.



(1.024M particles, 64x64 modes) Using fully symplectic space charge tracking is expected to reduce spurious numerical emittance growth [8].

Conclusion

New numerical tools to improve modeling of nonlinear integrable optics in IOTA with intense space charge have been implemented in the code IMPACT-Z. A complex treatment of the nonlinear insert is used as an alternative to [1] for tracking, avoiding a previously problematic instability. This is performed using a second-order symplectic integrator based on Yoshida splitting. Space charge can be treated using either a traditional grid-based Poisson solve or using a new spectral solver that is symplectic (by design) on the N-particle phase space of the macroparticle system [8]. Ongoing work will investigate how diffusion rates for the invariants of motion in IOTA are affected by choice of numerical parameters (number of macroparticles, number of grid cells or spectral modes, number of steps), and the use of additional Poisson solvers is under consideration.

References

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Acknowledgments

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